UDC 532.516

# ON OSCILLATORY INSTABILITY OF PLANE-PARALLEL CONVECTIVE MOTION IN A VERTICAL CHANNEL

PMM Vol. 36, №4, 1972, pp. 745-748

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(Received July 13, 1971)

A detailed quantitative study of the stability of a steady convective flow in a vertical channel is performed. The Bubnov-Galerkin method is used in high order approximations.

Papers [1 – 4] deal in detail with the perturbation spectra and the stability of a steady convective motion between parallel vertical surfaces heated to different temperatures. Solution of the spectral problem for the amplitudes of normal perturbations was performed on a digital computer using the Bubnov-Galerkin method, to show that the plane-parallel convective motion is unstable over a wide range of values of the Prandtl number $(0 < P < 10)$ with respect to monotonous perturbations which form a system of stationary vortices at the boundary dividing the counter-current flows. The critical Grashof number which determines the threshold of instability does not vary much over the range indicated for the Prandtl number, and this is related to the hydrodynamic character of the crisis.

The problem of oscillatory instability in a convective motion was already investigated in [5, 6], where the simplest approximations of the Bubnov-Galerkin method were used. The more recent calculations [2] show that the method converges very slowly when applied to the oscillatory branches of the spectrum, and the quantitative results obtained in [5, 6] for the oscillatory instability are not confirmed by the higher order approximations. The results of [2] imply quite confidently that, in particular, the crisis is related to the monotonous perturbations when $P < 10$.

Let us consider a vertical fluid layer bounded by infinite parallel surfaces $x = \mp h$, at which constant temperatures $\pm\Theta$ are maintained. We use the dimensionless variables based on unit distance, time, velocity and temperature and denoted by $h$, $h^2 / \nu$, $g\beta\Theta h^2 / \nu$ and $\Theta$, respectively, (where $\nu$ is the kinematic viscosity, $g$ is acceleration due to gravity and $\beta$ is the thermal expansion ratio. Using these variables we can express the velocity and temperature profiles of a closed, steady, plane-parallel flow in the following form:

$$v_0 = {}^1/_6 \ (x^3 - x), \qquad T_0 = -x \tag{1}$$

Examination of the stability of the steady mode (1) under small normal perturbations lead to the following boundary value problem for the amplitudes:

$$\begin{gathered} \Delta\Delta\varphi + ikGH\varphi + \theta' = -\lambda\Delta\varphi \\ P^{-1}\Delta\theta + ikG\,(T_0'\varphi - v_0\theta) = -\lambda\theta \\ \varphi = \varphi' = \theta = 0 \quad \text{for } x = \pm 1 \end{gathered} \tag{2}$$

$$\Delta = \partial^2 / \partial x^2 - k^2, \qquad H\varphi = v_0''\varphi - v_0\Delta\varphi$$
$$G = g\beta\Theta h^3 / \nu^2, \qquad P = \nu / \chi$$

Here $\varphi$ and $\theta$ are the amplitudes of the perturbations of the stream function and the temperature, $k$ and $\lambda$ are the wave number and the decrement, $G$ and $P$ are the Grashof and the Prandtl numbers, respectively.

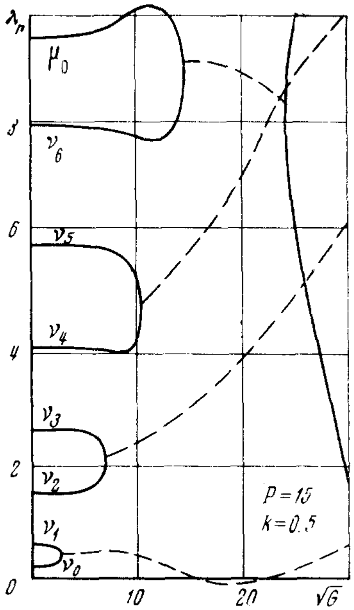


Fig. 1

As in [2 – 4], the amplitude problem was solved using the Bubnov-Galerkin method. The solution was constructed in the form of a superposition

$$\theta = \sum_{m=0}^{M-1} \alpha_m \theta_m, \qquad \varphi = \sum_{n=0}^{N-1} \beta_n \varphi_n \qquad (3)$$

where the amplitudes of the normal perturbations of the stationary fluid layer were chosen as the basis functions $\theta_m$ and $\varphi_n$. Expansions contain up to 28 basis functions ($M = N = 14$). The check for convergence was performed by comparing the results obtained with 16, 20 and 24 basis functions. The numerical data obtained in these approximations practically coincided. Figures 1 – 4 illustrate the main results of the computations. Figure 1 gives an example of the spectrum of decrements ($P = 15$; $k = 0.5$). We can see that with the increasing Grashof number the real levels merge pairwise forming complex conjugate pairs (their common real parts are represented by the broken lines).

The pair $(\nu_0, \mu_0)$ of levels decomposes with increasing $G$ back into a pair of real levels, one of which intersects the $G$-axis and generates a monotonous type instability. In addition to this instability which has a hydrodynamic character (the lowest hydrodynamic perturbation $\mu_0$ takes part in its creation), we also have an oscillatory type instability connected with two lowest thermal perturbations $\nu_0$ and $\nu_1$. The latter form a pair of oscillatory perturbations and the common real part of the decrements is negative in some interval

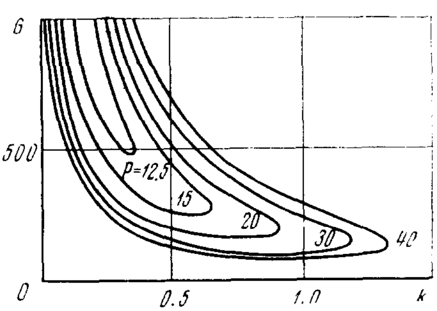


Fig. 2

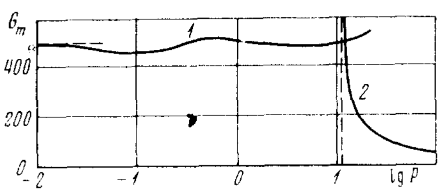


Fig. 3

of $G$. These perturbations represent thermal waves propagating along the stream. It is important to emphasise that the growth of the thermal perturbations is essentially determined by their interaction with the hydrodynamic perturbations. When the latter are absent, the thermal waves always decay [7].

The neutral curves of the oscillatory instability are shown in Fig. 2. It is interesting to note that for the given value of $k$ the region of instability is bounded from above in the direction of $G$ The oscillatory instability appears at fairly large Prandtl numbers $P > P_*$. Calculations indicate that $P_* = 11.4$. In the region $P > P_*$ the minimum critical value of the Grashof number $G_m$ decreases monotonously with increasing $P$. At fairly large $P$ we have the following asymptotic relation (*):

$$G_m = 470 / \sqrt{P} \qquad (4)$$

This formula in particular implies that the critical temperature difference increases with increasing viscosity according to the law $\theta \approx \nu^{1/2}$.

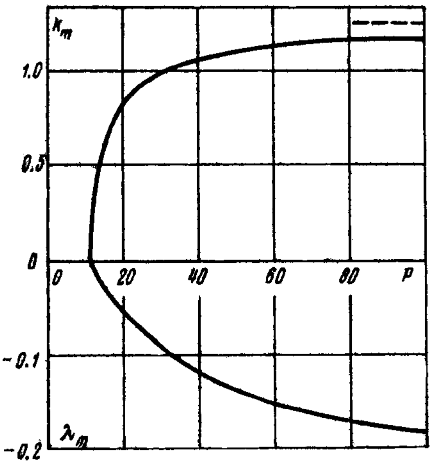


Fig. 4

Figure 3 depicts the results referring to the stability of a steady convective flow between vertical isothermal surfaces. Curve *1* gives the boundary of stability with respect to the monotonous perturbations and curve *2*, with respect to the oscillatory perturbations. At $P > 12$ the oscillatory perturbations are more dangerous. Instability of the high viscosity fluids is thus connected with the growth of the moving thermal waves.

The critical wave number $k_m$ of the oscillatory perturbations increases monotonously with increasing Prandtl number and tends to the limiting value of 1.25 (Fig. 4). The critical wave number is of the order of unity everywhere except a narrow region near $P_*$, i.e. just as in the case of monotonous instability, the characteristic dimension of the critical oscillatory perturbations is of order of the layer width.

The frequency and the phase velocity of the oscillatory perturbations are defined by the imaginary part of the decrement $\lambda_i$ The oscillatory instability is generated by a pair of complex conjugate perturbations (a "mixture" of $v_0$ and $v_1$) whose imaginary parts differ in sign. Therefore growing thermal waves are feasible, travelling in both upward and downward direction (since the stationary velocity profile is odd with respect to the middle of the channel, the waves may be carried by either an ascending or a descending convective flow).

The phase velocity can be conveniently compared with the maximum velocity of the steady flow equal (in the dimensional units) to $v_m = 0.0641\ g\beta\,\theta\,h^2/\nu$. Then the relative phase velocity (in the units of $v_m$) is equal, for the critical perturbation, to $u = 15.6\lambda_i/(k_m G_m)$. With increasing the Prandtl number from 20 to 100, the relative phase velocity increases monotonously from 0.93 to 1.02. Thus the neutral thermal waves propagate at the phase velocity almost equal to the maximum velocity of the steady flow.

The problem of oscillatory instability was also studied in [8] with the help of an asymptotic method based on expanding the solution into a power series in small parameter

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*) The approximations used in [6] and containing four basis functions give $P_* = 1.8$ and the value of the coefficient appearing in (4) is equal to 214.

$P^{-1/2}$. The principal term of the asymptotic expansion yields the limiting values of $k_m$= 1.25 and $u$ = 1.06, which agree with the results given above. The authors of [8] give also the limiting relationship $G_m(P)$, which differs from (4) in the value of the numerical factor (580 instead of 470).

In conclusion, it should be stressed that the experimental observations of oscillatory instability must be performed on channels of sufficient length, since a running thermal wave cannot appear in a short closed channel. Another restriction on the channel length is connected with the relatively slow growth of the oscillatory perturbations. The rate of growth can be characterized by the maximum (numerical) value of the real part of the decrement in the region of instability (Fig.1). The extremal (in $G$) value $\lambda_m$ corresponding to $k = k_m$ versus the Prandtl number, is shown in Fig.4. The data presented in Figs.1 and 4 indicate that the oscillatory perturbations grow at much slower rate than the monotonous ones. It is evident that oscillatory instability can only be observed when the exponential growth time $1/\lambda_r$ is less than the time $L/u$ ($L$ is the channel length and $u$ is the phase velocity) in which the wave travels the length of the channel. For the values of parameters $P = 40$, $k = 1$ and $G = 160$, the estimate yields $L/h > 70$.

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Translated by L.K.